

DISCRETE FOURIER RESTRICTION THEOREMS IN TWO DIMENSIONS.

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ABSTRACT. Consider the group \mathbb{R}^2 with the discrete topology, and denote its Fourier algebra by $A(\mathbb{R}_d^2)$. We reformulate a theorem of V.A. Yudin as a statement about restrictions of functions in $A(\mathbb{R}_d^2)$ to the boundary of a strictly convex domain when those functions vanish outside that boundary. We give visual proofs of that statement and a complementary one.

1. INTRODUCTION

Yudin's theorem [14] is about the Fourier coefficients, $\hat{f}(\vec{n})$ say, of an integrable function f on the product $\mathbb{T} \times \mathbb{T}$ of two copies of the unit circle group \mathbb{T} . Those coefficients are defined on the product $\mathbb{Z} \times \mathbb{Z}$ of two copies of the integer group \mathbb{Z} . He used a dual method to estimate the ℓ^2 norm of their restriction to the integer lattice points in the boundary of a strictly convex domain in \mathbb{R}^2 when \hat{f} vanishes outside that boundary. We give direct proofs of that estimate and of the corresponding estimate when \hat{f} vanishes inside the boundary.

As usual,

$$\hat{f}(n_1, n_2) = \left(\frac{1}{2\pi} \right)^2 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(t_1, t_2) e^{-n_1 t_1} e^{-n_2 t_2} dt_1 dt_2.$$

Use the same measure $(1/2\pi)^2 dt_1 dt_2$ in computing L^p norms. Given a subset D of \mathbb{R}^2 , denote its interior by $\text{Int}(D)$, its complement by D^c and its boundary by Γ .

Our main goal in this paper is to give visual proofs of both parts of an extension of the following statement.

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Theorem 1.1. *There is a constant C so that if D is a strictly convex set in \mathbb{R}^2 with boundary Γ , and if $f \in L^1(\mathbb{T}^2)$, then the estimate*

$$(1.1) \quad \left[\sum_{\vec{n} \in \Gamma \cap \mathbb{Z}^2} |\hat{f}(\vec{n})|^2 \right]^{1/2} \leq C \|f\|_1$$

follows from either of the following conditions:

- (1) \hat{f} vanishes on $\text{Int}(D) \cap \mathbb{Z}^2$.
- (2) \hat{f} vanishes on $\text{Int}(D^c) \cap \mathbb{Z}^2$.

Call these the *interior* and *exterior* cases. As in [14, p. 861], no uniform estimate of the form (1.1) is possible in either case for a family of sets D whose boundaries contain arbitrarily long arithmetic progressions in the integer lattice \mathbb{Z}^2 .

The validity of inequality (1.1) in the exterior case is Yudin's theorem; we give a new proof of it in Section 4. The fact that the inequality also holds in the interior case seems to be new; we prove it in a direct way in Section 4, and outline a dual proof in Section 6. We explain in Section 2 how both cases have single-variable precedents in Yves Meyer's paper [8] and related work. We describe the common part of our direct proofs of the two cases in Section 3, and discuss refinements of those methods in Section 5. In an appendix, we outline proofs of two known lemmas that we use throughout the paper.

The restriction theorem above applies to a subspace of $L^1(\mathbb{T}^2)$ defined by requiring that some Fourier coefficients vanish. Related conclusions hold [2], [15, Theorem 1] without the latter requirement when $L^1(\mathbb{T}^2)$ is replaced by $L^p(\mathbb{T}^2)$, where $p \geq 4/3$. Unlike most Fourier restriction theorems, that result and ours give global ℓ^2 estimates rather than local L^2 estimates.

In Section 7, we consider examples where our methods also yield ℓ^2 estimates on suitable subsets of shifted copies of Γ . These sometimes lead to global L^2 estimates of the following kind.

Example 1.2. Let $D = \{(u, v) \in \mathbb{R}^2 : v > u^2\}$, let $f \in L^1(\mathbb{R}^2)$, and let $k \in \mathbb{R}$. If \hat{f} vanishes on $\text{Int}(D)$ or $\text{Int}(D^c)$, then

$$(1.2) \quad \left[\int_{-\infty}^{\infty} |\hat{f}(u, u^2 + k)|^2 du \right]^{1/2} \leq C' |k|^{1/4} \|f\|_1.$$

2. CONTAGION OF WEAKNESS OF SIZE IN FOURIER ALGEBRAS

The standard notation for the set of Fourier coefficients of functions in $L^1(\mathbb{T}^2)$ is $A(\mathbb{Z}^2)$. This set is a Banach algebra under pointwise operations because $L^1(\mathbb{T}^2)$ is a Banach algebra under convolution. The

norm of \hat{f} in $A(\mathbb{Z}^2)$ is defined to be $\|f\|_1$. Denote the restriction of \hat{f} to a set S by $\hat{f}|_S$, and rewrite inequality (1.1) in the form

$$(2.1) \quad \left\| \hat{f}|_{(\Gamma \cap \mathbb{Z}^2)} \right\|_2 \leq C \|\hat{f}\|_{A(\mathbb{Z}^2)}.$$

Also view $A(\mathbb{Z}^2)$ as the set of sequences on \mathbb{Z}^2 that factor as convolution products of sequences in $\ell^2(\mathbb{Z}^2)$; this corresponds to the fact that $L^1 = L^2 \cdot L^2$ pointwise. Moreover, $\|\hat{f}\|_{A(\mathbb{Z}^2)}$ is the infimum of the products $\|g\|_2 \|h\|_2$ over all pairs (g, h) of sequences on \mathbb{Z}^2 for which $g * h = \hat{f}$. Given such a convolution factorization of \hat{f} , extend those factors the discrete group \mathbb{R}_d^2 by letting them vanish off \mathbb{Z}^2 . The corresponding extension of \hat{f} belongs to $A(\mathbb{R}_d^2)$, with a norm that is clearly no larger than the norm of \hat{f} in $A(\mathbb{Z}^2)$.

Theorem 1.1 follows immediately from the next statement.

Theorem 2.1. *There is a constant C so that if D is a strictly convex set in \mathbb{R}^2 with boundary Γ , and if $w \in A(\mathbb{R}_d^2)$, then the estimate*

$$(2.2) \quad \|w|_\Gamma\|_2 \leq C \|w\|_{A(\mathbb{R}_d^2)}$$

follows from either of the following conditions:

- (1) *w vanishes on $\text{Int}(D)$.*
- (2) *w vanishes on $\text{Int}(D^c)$.*

Here we use the notion of “boundary” in the usual topology on \mathbb{R}^2 . This makes the corresponding statement for the space $A(\mathbb{R}^2)$ true but trivial, because functions in $A(\mathbb{R}^2)$ are continuous relative to the usual topology on \mathbb{R}^2 , and they vanish on Γ if they do so on $\text{Int}(D)$ or $\text{Int}(D^c)$.

Meyer’s result [8, pp. 532–533] on \mathbb{Z} extends to \mathbb{R}_d as follows.

Theorem 2.2. *Let $(x_j)_{j=1}^J$ be a sequence of positive numbers satisfying the condition that $x_{j+1} \geq (1 + \delta)x_j$ for some positive constant δ and all j . Let $w \in A(\mathbb{R}_d)$. Then an estimate*

$$(2.3) \quad \left[\sum_{j=1}^J |w(x_j)|^2 \right]^{1/2} \leq C(\delta) \|w\|_{A(\mathbb{R}_d)}.$$

follows from either of the following conditions:

- (1) *w vanishes on each of the intervals $(x_j/(1 + \delta), x_j)$.*
- (2) *w vanishes on each of the intervals $(x_j, (1 + \delta)x_j)$.*

We will not prove this here, but we note that, as in [4], the first part, about coefficients after long-enough gaps, follows by the method that we use to prove the first part of Theorem 2.1. As in [5, page 214], the second part above follows from Remark 5.3 below.

Meyer used other methods to prove the version of Theorem 2.2 for $A(\mathbb{Z})$. He described the pattern in his theorem as a “contagion of weakness of size.” On any infinite discrete abelian group G_d , use the ℓ^2 norm to measure this weakness, noting that $\|w\|_{A(G_d)} \leq \|w\|_2$, and recalling that the most one generally say about the size of a function in $A(G_d)$ is that it belongs to $c_0(G_d)$, which strictly includes $\ell^2(G_d)$.

Denote the indicator function of a set S by 1_S . If $w|_{\text{Int}(D)}$ belongs to $\ell^2(D)$, then applying the first part of Theorem 2.1 to $w - w \cdot 1_{\text{Int}(D)}$ yields that

$$(2.4) \quad \|w|_{\Gamma}\|_2 \leq C\|w\|_{A(\mathbb{R}_d^2)} + C\|w|_{\text{Int}(D)}\|_2.$$

Similarly, if $w|_{\text{Int}(D^c)}$ belongs to $\ell^2(D^c)$, then

$$(2.5) \quad \|w|_{\Gamma}\|_2 \leq C\|w\|_{A(\mathbb{R}_d^2)} + C\|w|_{\text{Int}(D^c)}\|_2.$$

In the setting of Theorem 2.2, replace $\text{Int}(D)$ or $\text{Int}(D^c)$ with the union of long-enough gaps ending or beginning at the numbers x_j . In each case, weakness of a member of $A(\mathbb{R}_d^2)$ or $A(\mathbb{R}_d)$ on a suitable set propagates to the boundary of that set in \mathbb{R}^2 or \mathbb{R} .

Remark 2.3. The methods for the second part of Theorem 2.1 can also be used [6, 5, 13] to prove Paley’s theorem about coefficients of functions in the classical space $H^1(\mathbb{T})$. In that setting, weakness on any Hadamard set of positive integers follows from weakness on the set \mathbb{Z}_- of negative integers. It is less clear how Hadamard sets in \mathbb{Z}_+ can be regarded as parts of some boundary of \mathbb{Z}_- . But they share with the strictly-convex examples the property that certain combinations of “boundary points” must belong to the set where weakness is assumed to occur. See Remark 5.1 for more on this.

Remark 2.4. Recall that \mathbb{R}_d is dual to the Bohr compactification $\text{b}\mathbb{R}$ of the real line. As in [8, page 534], applying standard duality arguments to Theorem 2.2 yields that if $(v_j) \in \ell^2$, then there exist functions G and H in $L^\infty(\text{b}\mathbb{R})$ with the following properties.

- (1) $\|G\|_\infty$ and $\|H\|_\infty$ are both no larger than $C(\delta)\|v\|_2$.
- (2) $\hat{G}(x_j) = \hat{H}(x_j) = v_j$ for all j .
- (3) \hat{G} vanishes outside the union of the intervals $(x_j/(1+\delta), x_j]$.
- (4) \hat{H} vanishes outside the union of the intervals $[x_j, (1+\delta)x_j]$.

If $x_{j+1} \geq (1+\delta)^2 x_j$ for all j , then the supports of \hat{G} and \hat{H} are disjoint except for the numbers x_j . Work by Goes [7, §4] exhibited similar patterns in a different context. As in [5, pp. 214–215], they yield an easy proof of the Grothendieck inequality, which follows in the same way from the duals of Theorem 1.1 and 2.1 that we discuss in Section 6.

3. TWO LEMMAS

In our proofs of the nontrivial cases of Theorem 2.1, we write each value of w as an inner product of one function in $\ell^2(\mathbb{R}_d^2)$ with a translate of another such function. Recall that for a function v on an additive abelian group and a point x in that group, the function $\tau_x v$ maps each point y to $v(y - x)$, and the function v^* maps each point y to $\overline{v(-y)}$. Rename the factor h in $w = g * h$ as h^* , with no effect on norms. Since

$$(3.1) \quad \begin{aligned} (g * h^*)(x) &= \sum_{y \in \mathbb{R}^2} g(y) h^*(x - y) = \sum_{y \in \mathbb{R}^2} g(y) \overline{h(y - x)}, \\ w(x) &= (g, \tau_x h). \end{aligned}$$

Proving Theorem 2.1 therefore reduces to bounding $\sum_{j=1}^J |(g, \tau_{x_j} h)|^2$ for finite sequences $(x_j)_{j=1}^J$ of distinct points in Γ .

We apply the lemmas below with $H = \ell^2(\mathbb{R}_d^2)$ and $A_j = \tau_{x_j}$. The first lemma goes back to [3], and led to a rediscovery [4] of Meyer's result about coefficients after gaps. The second lemma is more recent [6], and was used there to reprove the extension [5, Theorem 2] of Paley's theorem that yields the part of Theorem 2.2 about coefficients before gaps. In the next section, we specify subspaces with the properties required in the lemmas. We outline proofs of the lemmas in Appendix A.

Lemma 3.1. *Let H be a Hilbert space and $M_1 \subset M_2 \subset \cdots \subset M_J$ be closed subspaces of H . Let A_1, A_2, \dots, A_J be unitary operators on H for which*

$$A_1 M_1 \subset A_2 M_2 \subset \cdots \subset A_J M_J.$$

Let g and h be members of H satisfying the following conditions for all indices $j < J$.

- (1) $A_j h \in A_{j+1} M_{j+1}$.
- (2) *The vector g is orthogonal to the subspace $A_{j+1} M_j$.*

Then

$$(3.2) \quad \left[\sum_{j=1}^J |(g, A_j h)|^2 \right]^{1/2} \leq 2(\|g\|_H) \|h\|_H.$$

Lemma 3.2. *Let H be a Hilbert space and $L_1 \supset L_2 \supset \cdots \supset L_J$ be closed subspaces of H . Let A_1, A_2, \dots, A_J be unitary operators on H for which*

$$A_2 L_1 \subset A_3 L_2 \subset \cdots \subset A_J L_{J-1}.$$

Let g and h be elements of H satisfying the following conditions:

- (1) $A_j h \in A_{j+1} L_j$ for all $j < J$.
- (2) *The vector g is orthogonal to the subspace $A_j L_j$ for all $j > 1$.*

Then

$$(3.3) \quad \left[\sum_{j=1}^J |(g, A_j h)|^2 \right]^{1/2} \leq 2(\|g\|_H) \|h\|_H.$$

4. VISUAL PROOFS

Given the convolution factorization $w = g * h^*$ and a subset E of \mathbb{R}_d^2 , let $V(E, h)$ denote the closure in $H = \ell^2(\mathbb{R}_d^2)$ of the subspace spanned by the translates $\tau_x h$ for which $x \in E$. In the interior case of Theorem 2.1, we will apply Lemma 3.1 with $M_j = V(E_j, h)$ for suitable sets E_j . In the exterior case, we will apply Lemma 3.2 with $L_j = V(D_j, h)$ for suitable sets D_j .

The nesting and membership conditions in Lemma 3.1 hold if

$$(4.1) \quad E_j \subset E_{j+1}, \quad x_j + E_j \subset x_{j+1} + E_{j+1}, \quad \text{and} \quad x_j \in x_{j+1} + E_{j+1}$$

for all $j < J$. The orthogonality condition holds if $(g, \tau_y h) = 0$ for all y in $x_{j+1} + E_j$. Equation (3.1) makes this equivalent to having $w(y) = 0$ for all such y .

Let $F_j = x_j + E_j$ for all j , and let $\Delta x_j = x_{j+1} - x_j$ when $j < J$. The last condition in the previous paragraph is equivalent to requiring that w vanish on all the sets $F_j + \Delta x_j$ with $j < J$. In the interior case, this happens if those sets are all included in $\text{Int}(D)$. Translate the other conditions on the sets E_j to see that it suffices in that case to find sets F_j satisfying the following four conditions for all $j < J$.

$$(4.2) \quad F_j + \Delta x_j \subset \text{Int}(D), \quad F_j + \Delta x_j \subset F_{j+1},$$

$$(4.3) \quad F_j \subset F_{j+1}, \quad \text{and} \quad x_j \in F_{j+1}.$$

That is,

$$(4.4) \quad F_j + \Delta x_j \subset \text{Int}(D) \cap F_{j+1} \quad \text{and} \quad F_j \cup \{x_j\} \subset F_{j+1}.$$

Call these the *shifted inclusions* and the *unshifted inclusions*.

Similarly, the subspaces L_j and their images $A_{j+1}L_j$ nest as prescribed in Lemma 3.2 if

$$D_1 \supset D_2 \supset \cdots \supset D_J,$$

$$\text{and} \quad x_2 + D_1 \subset x_3 + D_2 \subset \cdots \subset x_J + D_{J-1}.$$

The membership condition in the lemma holds if $x_j \in x_{j+1} + D_j$ for all $j < J$, and the orthogonality condition holds in the exterior case if $x_j + D_j \subset \text{Int}(D^c)$ for all $j > 1$.

Consider the sets $G_{j+1} = x_{j+1} + D_j$, creating another point x_{J+1} to cover the case where $j = J$. Translate the conditions on the sets D_j to

see that it suffices that

$$(4.5) \quad G_{j+1} - \Delta x_j \subset \text{Int}(D^c), \quad G_{j+1} - \Delta x_j \subset G_j,$$

$$(4.6) \quad G_j \subset G_{j+1}, \quad \text{and} \quad x_j \in G_{j+1}.$$

In this case, the shifted inclusions and unshifted inclusions state that

$$(4.7) \quad G_{j+1} - \Delta x_j \subset \text{Int}(D^c) \cap G_j \quad \text{and} \quad G_j \cup \{x_j\} \subset G_{j+1}.$$

If the boundary Γ of D is the graph of a strictly convex or strictly concave function defined on all of \mathbb{R} , and the points x_j run from left to right along Γ , then we can use sets F_j and G_j that are very similar. For such a concave function ϕ , write $x_j = (u_j, v_j)$, and

$$(4.8) \quad \text{let} \quad F_j = \{(u, v) \in \mathbb{R}^2 : u < u_j \text{ and } v \leq \phi(u)\},$$

$$(4.9) \quad \text{while} \quad G_j = \{(u, v) \in \mathbb{R}^2 : u < u_j \text{ and } v \geq \phi(u)\}.$$

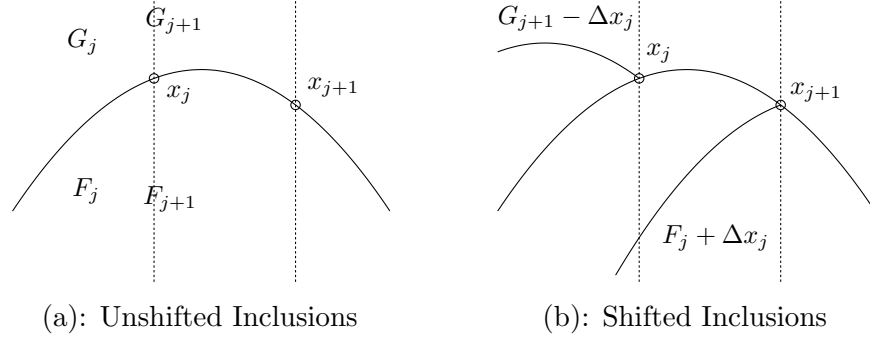


FIGURE 1. Similar Sets

The unshifted inclusions clearly hold for both F_j and G_j . By strict concavity, any part of the boundary ending at x_j rises strictly more rapidly or falls strictly more slowly than any part of the same width to the right of it. Shifting such a part ending at x_j by Δx_j gives a curve that ends at x_{j+1} and lies strictly below Γ except at x_{j+1} . This yields the shifted inclusions for the sets $F_j + \Delta x_j$. The corresponding inclusions for the sets $G_{j+1} - \Delta x_j$ follow in a similar way. Both cases of Theorem 2.1 therefore hold with $C = 2$ for such sets D .

Every unbounded, strictly convex set can be rotated to have the form specified above, except that the domain of the function ϕ may not be all of \mathbb{R} . In that case, add the requirement that u belong to the domain of ϕ in defining F_j . If the domain of ϕ is bounded on the left, also include all vertical lines to the left of D in defining G_j .

When D is bounded and strictly convex, follow [14] in recalling that there are vertical support lines at two boundary points, listed from left to right as x_0 and x_∞ say. In the exterior case, let Γ_0 be the upper boundary with x_∞ excluded.

Consider points x_j running from left to right in Γ_0 , starting with x_0 . As above, let G_j consist of all points in \mathbb{R}^2 that lie strictly to the left of x_j , and that do not lie directly below Γ_0 . Then the inclusions (4.7) hold for all $j \geq 0$, so that $\|w|_{\Gamma_0}\|_2 \leq 2\|w\|_{A(\mathbb{R}_d^2)}$. Rotate by 180° to get a similar estimate on the rest of Γ , and that

$$(4.10) \quad \|w|_{\Gamma}\|_2 \leq 2\sqrt{2}\|w\|_{A(\mathbb{R}_d^2)}.$$

In the interior case for the same set D , shear vertically and shift to place both of the points x_0 and x_∞ on the u -axis; this does not affect $\|w\|_{A(\mathbb{R}_d^2)}$. Then the lower boundary lies below the u -axis. There will be one point, x_J say, on the upper boundary with a horizontal support line. Place that point on the v -axis. Then the upper boundary in the second quadrant is the graph of an increasing function.

Consider points $\{x_j\}_{j=1}^{J-1}$ running from left to right in the interior of that graph. Find the midpoint of the line segment from x_0 to x_j ; then rotate the part of boundary curve running from x_0 to x_j by 180° about that midpoint to get a lower curve returning to x_0 from x_j . Form the convex hull of that lower curve and the upper boundary curve from x_0 to x_j , and delete the vertices x_0 and x_j to get the set F_j . Form F_J in the same way. We show this in Figure 2(a) below.

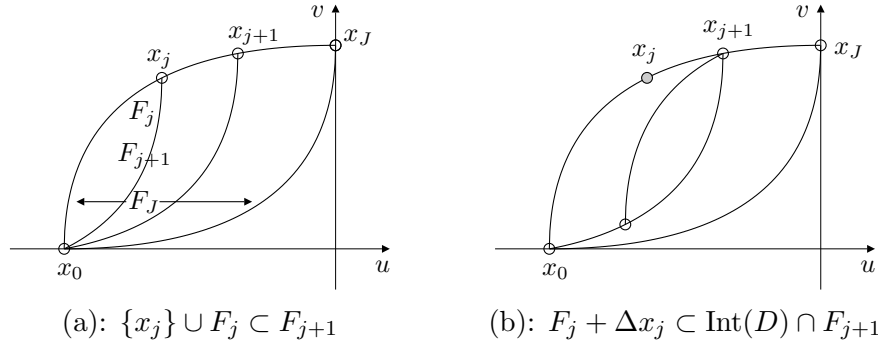


FIGURE 2. Interior Inclusions

It is obvious that $\{x_j\} \subset F_{j+1}$. The sets F_j and F_{j+1} are mapped onto themselves by the 180° rotations, ψ_j and ψ_{j+1} say, about their centroids. Note that ψ_{j+1} is equal to ψ_j followed by the shift by Δx_j ; so ψ_{j+1} maps F_j onto $F_j + \Delta x_j$. Since the upper boundary of F_j is, by definition, an initial part of the upper boundary of F_{j+1} , the lower

boundary of $F_j + \Delta x_j = \psi_{j+1}(F_j)$ is a final part of the lower boundary of F_{j+1} .

As in Figure 1(b), the upper boundary of $F_j + \Delta x_j$ lies strictly below the upper boundary of F_{j+1} except at the missing point x_{j+1} . Hence $F_j + \Delta x_j \subset F_{j+1}$; applying ψ_{j+1} again then makes $F_j \subset F_{j+1}$.

The lower boundary of F_J runs from x_0 to x_J , and is the graph of an increasing function. Hence F_J lies strictly inside the second quadrant, as do its subsets $F_j + \Delta x_j$ with $j < J$. These sets therefore do not meet the lower boundary or right-hand boundary of D . Since the shifted sets $F_j + \Delta x_j$ lie strictly below the upper boundary of D in the second quadrant, they are included in $\text{Int}(D)$, as required.

Let Γ_2 be the part of Γ inside the second quadrant, together with x_J . Then $\|w|_{\Gamma_2}\|_2 \leq 2\|w\|_{A(\mathbb{R}_d^2)}$ in the interior case. Similar arguments on three other parts of Γ yield that

$$(4.11) \quad \|w|_{\Gamma}\|_2 \leq 4\|w\|_{A(\mathbb{R}_d^2)}.$$

5. WEAKER HYPOTHESES

Our methods work when w vanishes on some sets that are smaller than the ones used in Section 4. In the next section, we discuss dual methods that also work with those weaker hypotheses.

Fix a finite sequence $(x_j)_{j=1}^J$. It will turn out to suffice that w vanish on suitable subsets of the additive group generated by the points x_j . All points x in that group have the form

$$(5.1) \quad x = \sum_{i=1}^J \varepsilon_i x_i,$$

where the coefficients ε_i are integers.

The application of Lemma 3.1 to lacunary Fourier series was analysed in [4, Remark 3]. In the present context, the same reasoning shows that it suffices for w to vanish on the set $\text{Alt}((x_j))$ of points x with alternating sum representations

$$(5.2) \quad x = x_{j_1} - x_{j_2} + \cdots + x_{j_{2i-1}} - x_{j_{2i}} + x_{j_{2i+1}}$$

with at least 3 terms and a strictly-increasing index sequence (j_ℓ) .

Let F_{j+1} be the set of points x as above with $j_{2i+1} \leq j+1$, but only impose the requirement that the sum (5.2) have at least 3 terms when $j_{2i+1} = j+1$. These sums belong to the fatter sets F_{j+1} shown in Figures 1(a) and 2(a). The inclusions (4.4) hold for the smaller sets F_j and F_{j+1} , and Lemma 3.1 applies.

For Lemma 3.2, the analysis in [6, Section 5] yields the sets G_{j+1} consisting of all points x with a representation

$$(5.3) \quad x = x_i - \sum_{j' \geq i} n_{j'} \Delta x_{j'},$$

satisfying the following conditions:

- (1) $i \leq j + 1$.
- (2) The coefficients $n_{j'}$ are nonnegative integers.
- (3) If $i = j + 1$, then $n_{j'} \neq 0$ for some j' .

The points in this version of G_{j+1} belong to the fatter set G_{j+1} shown in Figure 1(a). The desired inclusions hold for the smaller sets G_j and G_{j+1} .

The lemma applies provided that w vanishes on the union $\text{Sch}((x_j))$ of the smaller sets $G_{j+1} - \Delta x_j$. The points x in that union are those with a representation (5.3) satisfying condition (2) with $n_{j'} \neq 0$ for some j' . They are also given by the sums of the form (5.1) where the integer coefficients ε_i have the following properties:

- The full sum $\sum_{i=1}^J \varepsilon_i$ is equal to 1.
- All partial sums of the full sum are nonnegative.
- All partial sums after the first positive one are positive.
- Some partial sum is greater than 1.

These conditions also arose in [5] and [14].

Remark 5.1. In the setting of Remark 2.3, Paley's theorem holds because the set $\text{Sch}(\{n_j\})$ is included in \mathbb{Z}_- when the sequence (n_j) is sufficiently lacunary. This was used in a dual way in [5] and [13], and in a direct way in [6].

Remark 5.2. We made one choice of the sets G_j in proving the exterior case of Theorem 2.1, and another just above. For both choices, the corresponding sets D_j are additive semigroups. This can be used [6, Remark 5.7] to define suitable partial orders on \mathbb{R}^2 , relating that case of Theorem 2.1 to Paley's theorem.

Remark 5.3. It can happen that $|\varepsilon_i| > 1$ in the sums (5.1) representing points in $\text{Sch}((x_j))$. Let $S((x_j))$ consist of all points x with representations (5.1) in which the coefficients ε_i belong to the set $\{-1, 0, 1\}$ and satisfy the four conditions for membership of x in $\text{Sch}((x_j))$. Arguments in [5] and [6] each combine with the application above of Lemma 3.2 to show that

$$(5.4) \quad \|w|X\|_2 \leq 4\|w\|_{A(\mathbb{R}_d^2)}$$

when w vanishes on $S((x_j))$.

Remark 5.4. For $\text{Alt}((x_j))$, rewrite the representation (5.2) in the form

$$(5.5) \quad x = x_{j_{2i+1}} - \sum_{j' < j_{2i+1}-1} n_{j'} \Delta x_{j'}$$

where the coefficients $n_{j'}$ take the values 0 and 1 only and the latter occurs at least once. For F_{j+1} , keep those conditions on $(n_{j'})$, put $j_{2i+1} = j + 1$, and require instead that $j' \leq j$ in the sum.

6. DUAL CONSTRUCTIONS

Denote the Bohr compactification of \mathbb{R}^2 by $\text{b}\mathbb{R}^2$. The duality arguments in [12] or [8, page 534] show that Theorem 2.1 is equivalent to the one below. Theorem 1.1 has a similar dual.

Theorem 6.1. *Let D be a strictly convex set in \mathbb{R}^2 with boundary Γ . Then for each function v in $\ell^2(\Gamma)$, there exist functions G and H in $L^\infty(\text{b}\mathbb{R}^2)$ with the following properties:*

- (1) $\|G\|_\infty$ and $\|H\|_\infty$ are both no larger than $C\|v\|_2$.
- (2) The restrictions of \hat{G} and \hat{H} to Γ both coincide with v .
- (3) \hat{G} vanishes on $\text{Int}(D^c)$.
- (4) \hat{H} vanishes on $\text{Int}(D)$.

Theorem 2.1 can be proved by constructing suitable functions G and H when the support of v is finite. Choose points x_j as in Section 4. Let v vanish off the set $X = \{x_j\}_{j=1}^J$, with $\|v\|_2 = 1$. The modification of the Rudin-Shapiro construction in [1] produces a trigonometric polynomial G with the following properties.

- $\|G\|_\infty \leq C$.
- $\hat{G}|_X = v$ if the sets X and $\text{Alt}((x_j))$ are disjoint.
- \hat{G} vanishes off the set $X \cup \text{Alt}((x_j))$.

This yields the first part of Theorem 2.1, since the strict convexity of the unbounded set D makes $\text{Alt}((x_j))$ a subset of $\text{Int}(D)$ in the diagrams in Section 4.

It also follows that $\text{Sch}((x_j)) \subset \text{Int}(D^c)$ in those cases. For the second part of the theorem, it suffices to construct a function H with the following properties.

- $\|H\|_\infty \leq 1$.
- $\overline{v(x_j)} \hat{H}(x_j) \geq (1/C)|v(x_j)|^2$ for all x_j .
- \hat{H} vanishes off the set $X \cup \text{Sch}((x_j))$.

Yudin refined a method of Pigno and Smith [10, 13] for this, and noted that a construction in [5] would work too. In both of these methods, one can satisfy the middle condition above by making $\hat{H}|_K = (1/C)v$.

7. SEPARATED POINTS IN SHIFTED CURVES

In Example 1.2, let $k > 0$ and $h > \sqrt{k/2}$. We will show that

$$(7.1) \quad \left[\sum_{j=-\infty}^{\infty} \left\{ \sup_{jh \leq u < (j+1)h} |\hat{f}(u, u^2 - k)| \right\}^2 \right]^{1/2} \leq \sqrt{2}C \|f\|_1,$$

in the interior case, and that

$$(7.2) \quad \left[\sum_{j=-\infty}^{\infty} \left\{ \sup_{jh \leq u < (j+1)h} |\hat{f}(u, u^2 + k)| \right\}^2 \right]^{1/2} \leq \sqrt{2}C \|f\|_1$$

in the exterior case. Inequality (1.2) then follows because

$$\int_{jh}^{(j+1)h} |g(u)|^2 du \leq h \left\{ \sup_{jh \leq u < (j+1)h} |g(u)| \right\}^2$$

for all measurable functions g .

The “amalgam norm” estimates (7.1) and (7.2) follow from ℓ^2 estimates on sets of suitably separated points, $x_j = (u_j, u_j^2)$ say, in Γ . Let $w \in A(\mathbb{R}^2)$ and require it to vanish that w vanishes on the region where $v > u^2 + k$, or on the region where $v < u^2 - k$. Then

$$(7.3) \quad \left[\sum_j |w(x_j)|^2 \right]^{1/2} \leq C \|w\|_{A(\mathbb{R}^2)} \quad \text{if } \Delta u_j > \sqrt{k/2} \text{ for all } j.$$

Apply this to shifted copies w of \hat{f} , and choose points x_j in alternate intervals $[j'h, (j'+1)h)$ to get the estimates (7.1) and (7.2).

In proving inequality (7.3), we consider more general sets D of the form $\{(u, v) : v \geq \phi(u)\}$, where $\phi''(u) \geq c > 0$. Our methods apply to $A(\mathbb{R}^2)$, and yield inequality (7.3) if the sets $\text{Alt}((x_j))$ and $\text{Sch}((x_j))$ are respectively included in the sets $\text{Int}(D) + (0, k)$ and $\text{Int}(D^c) - (0, k)$.

Given a point x in $\text{Sch}((x_j))$ in the form (5.3), let $n = \sum_{j'} n_{j'}$ and say that x is an n -th generation descendant of x_i . Subtracting another copy of Δx_j , where $j \geq i$, from x gives an $(n+1)$ -st descendant, x' say. All descendants (u, v) of x_i share the property that $u < u_i$. Visual arguments in the style of Section 4 show that if ϕ is strictly convex and $x \in \text{Int}(D^c - (0, k))$, then $x' \in \text{Int}(D^c - (0, k))$ too.

So it suffices to check that first-generation points in $\text{Sch}((x_j))$ belong to $\text{Int}(D^c - (0, k))$. They have the form $x_i - \Delta x_j$ where $j \geq i$. Rewriting this as $(u, v) = (u_i, v_i) - (\Delta u_j, \Delta v_j)$ reduces matters to showing

that $\phi(u) - v > k$. Now

$$\begin{aligned} v_i &= \phi(u_i) = \phi(u) + \int_u^{u_i} \phi'(r) dr, \quad \phi(u) = v_i - \int_0^{\Delta u_j} \phi'(u+s) ds, \\ \Delta v_j &= \int_{u_j}^{u_{j+1}} \phi'(r) dr, \quad \text{and} \quad v = v_i - \Delta v_j = v_i - \int_0^{\Delta u_j} \phi'(u_j+s) ds. \end{aligned}$$

Therefore,

$$\begin{aligned} \phi(u) - v &= \int_0^{\Delta u_j} [\phi'(u_j+s) - \phi'(u+s)] ds \\ &= \int_0^{\Delta u_j} \left[\int_u^{u_j} \phi''(t+s) dt \right] ds \geq c(\Delta u_j)^2. \end{aligned}$$

Use the representation (5.5) to introduce a similar notion of generations of descendants in $\text{Alt}((x_j))$, but add the requirement that the extra nonzero coefficient $n_{j'}$ for the child $x' = (u', v')$ occurs before all nonzero coefficients for the parent x . Rename j_{2i+1} as $j+1$; then $u' \geq u_j + \Delta u_j$. Argue visually to reduce matters to first-generation cases where

$$x' = (u', v') = (u_{j+1}, v_{j+1}) - (\Delta u_{j'}, \Delta v_{j'}), \quad \text{and} \quad j' < j.$$

As above,

$$v' - \phi(u') = \int_0^{\Delta u_{j'}} \left[\int_{u_{j'}}^{u'} \phi''(t+s) dt \right] ds \geq c(\Delta u_{j'}) \Delta u_j.$$

The inclusions $\text{Sch}((x_j)) \subset \text{Int}(D^c - (0, k))$ and $\text{Alt}((x_j)) \subset \text{Int}(D + (0, k))$ follow if $\Delta u_j > \sqrt{k/c}$ for all j .

The outcome changes if the graph of ϕ has an asymptote.

Example 7.1. Let $D_\alpha = \{(u, v) : u > 0, v > u^{-\alpha}\}$, where α is a positive constant. Let $f \in L^1(\mathbb{R}^2)$, and let $k > 0$. If \hat{f} vanishes on $\text{Int}(D_\alpha^c)$, then

$$(7.4) \quad \left[\int_0^\infty \left| \hat{f}(u, u^{-\alpha} + k) \right|^2 \frac{du}{u} \right]^{1/2} \leq C \|f\|_1.$$

There are cases where $f \in L^1(\mathbb{R}^2)$ and \hat{f} vanishes on $\text{Int}(D_\alpha)$ but

$$\int_0^\infty \left| \hat{f}(u, u^{-\alpha} - k) \right|^2 \frac{du}{u} = \infty.$$

The positive result here follows from the extension of Paley's inequality to functions f in $L^1(\mathbb{R}^2)$ for which $\hat{f}(u, v) = 0$ on the “negative”

semigroup, $-P$ say, where $u \leq 0$ and $v < 0$ if $u = 0$. That extension gives an ℓ^2 estimate for $(\hat{f}(x_j))$ when the sequence (x_j) satisfies the Hadamard condition that $2x_j - x_{j+1} \in -P$ for all j . So do the appropriate methods in Sections 5 or 6. These approaches all show that

$$\left[\int_0^\infty \sup_{v \in \mathbb{R}} \left| \hat{f}(u, v) \right|^2 \frac{du}{u} \right]^{1/2} \leq C \|f\|_1.$$

To get the negative results, use the fact that for each parallelogram, B say, with positive area, there is a function in the unit ball of $A(\mathbb{R}^2)$ that vanishes outside B and that exceeds $1/4$ on $1/4$ of the area of B . One way to confirm this fact runs via the argument applied to arithmetic progressions in [14, p. 861].

Similar reasoning, going back to [12], shows that if a nonnegative measure ν has the property that

$$\int_0^\infty \left| \hat{f}(u, u^{-\alpha} + k) \right|^2 d\nu(u) < \infty$$

whenever $f \in L^1(\mathbb{R}^2)$ and \hat{f} vanishes on $\text{Int}(D_\alpha^c)$, then

$$\nu((2^j, 2^{j+1}]) \leq C' \quad \text{for all } j.$$

Remark 7.2. Affine arclength measure is prominent in restriction theorems [9] for transforms of functions in $L^p(\mathbb{R}^2)$ when $p > 1$. The measure du on the graphs of $v = u^2 \pm k$ is affine invariant, but the measure du/u on the graph of $v = \phi_\alpha(u) + k$ is not, except when $\alpha = 1$.

APPENDIX A. TWO ORTHOGONALITY STEPS

We prove both lemmas by splitting the sequence $(g, A_j h)_{j=1}^J$ as a sum of two sequences whose ℓ^2 norms are easy to bound.

In Lemma 3.2, let P_j and Q_j be the orthogonal projections onto the subspaces L_j and $A_{j+1}L_j$ respectively, with $j < J$ in the latter case. Also let $Q_J = I$ and $Q_0 = 0$. By the membership condition in the lemma,

$$(A.1) \quad (g, A_j h) = (g, Q_j A_j h) = (Q_j g, A_j h) = a_j + b_j,$$

where $a_j = ((Q_j - Q_{j-1})g, A_j h)$ and $b_j = (Q_{j-1}g, A_j h)$ for all j . Then $b_1 = (Q_0 g, A_1 h) = 0$, and $b_j = (g, A_j(P_{j-1} - P_j)h)$ when $j > 1$, since $A_j P_{j-1} = Q_{j-1} A_j$ and $(g, A_j P_j h) = 0$ in that case. The projections $Q_j - Q_{j-1}$ have mutually orthogonal ranges, as do the projections $P_{j-1} - P_j$. By Cauchy-Schwarz, $\|(a_j)\|_2$ and $\|(b_j)\|_2$ are both bounded above by $(\|g\|_H)\|h\|_H$, and inequality (3.3) follows.

In Lemma 3.1, consider the orthogonal projections P_j and Q_j onto the subspaces M_j and $A_j M_j$. Also let $Q_{J+1} = I$ and $P_0 = 0$. This time, $(g, A_j h) = (Q_{j+1} g, A_j h)$, which splits as

$$(A.2) \quad ((Q_{j+1} - Q_j)g, A_j h) + (g, A_j(P_j - P_{j-1})h),$$

since $(g, A_j P_{j-1} h) = 0$ and $(g, A_j P_j h) = (Q_j g, A_j h)$. Finish as above.

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